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Structure of the Primary Hamiltonian Constraints for a Singular N-Body Lagrangian

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The structure of the primary Hamiltonian constraints is examined for a simple singular Lagrangian form (N interacting particles). The problem is a straightforward generalization of well-known two-particle cases.

1. INTRODUCTION AND NOTATION

In a given frame of reference Σ we assume there are $N \geq 1$ point particles; that is, a total of $4N$ "position" coordinates

$$x^{\mu}_{(i)}, \qquad \mu = 0, 1, 2, 3 \text{ (axis label)} \qquad i = 1, 2, \ldots, N \text{ (particle label)} \tag{1}$$

We define the generalized coordinates

$$q^{\mu}_{j} = \alpha^{i}_{j} x^{\mu}_{(i)}; \qquad i, j = 1, 2, \ldots, N \tag{2}$$

where (α^{i}_{j}) is a constant nonsingular matrix. (Note the summation of repeated up and down indices.) For a simpler notation, we also introduce the symbol $Q = \{Q^A\}$, $A = 1, 2, \ldots, 4N$, defined by

$$\{Q^A\} = \{q^{\mu}_{j}\} \tag{3}$$

For example, one can choose

$$Q^1 = q^0_1, \ldots, \quad Q^4 = q^3_1 \qquad Q^{4N-3} = q^0_N, \ldots, \quad Q^{4N} = q^3_N \tag{4}$$

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2. LAGRANGIAN AND CONSTRAINTS

Assuming that the (isolated) system can be described with respect to an evolution parameter τ (Kalb and Van Alstine, 1976; Raspini, 1983), we now consider a reparametrization-invariant Lagrangian L (Kalb and Van Alstine, 1976; Bolza, 1904). The simplest form, modeled on the relativistic single-particle case, is clearly

$$\begin{aligned} L &= -[\dot{Q}^A G_{AB}(Q) \dot{Q}^B]^{1/2}; \qquad G_{AB} = G_{BA} \\ \dot{Q}^A &= \frac{dQ^A}{d\tau}; \qquad A, B = 1, 2, \ldots, 4N \end{aligned} \tag{5}$$

The Lagrangian in (5) is singular, and the corresponding Legendre Hamiltonian is identically vanishing (Kalb and Van Alstine, 1976; Bolza, 1904). A Dirac-type treatment must then be applied (Kalb and Van Alstine, 1976; Raspini, 1983) if one wishes to obtain an appropriate Hamiltonian formulation. Of course, before starting this approach, one should restrict the $(\dot{Q}, Q)$ space according to

$$\dot{Q}^A G_{AB} \dot{Q}^B > 0 \tag{6}$$

which is the regularity condition for L.

From L, we define the P canonical momenta:

$$P_A = -\partial L / \partial \dot{Q}^A = G_{AB} S^B \tag{7a}$$

$$S^B = -\dot{Q}^B / L \tag{7b}$$

Momenta are not all independent, and some "primary constraints" exist among them (Kalb and Van Alstine, 1976, Raspini, 1983; Dirac, 1964; Shanmugadhasan, 1973; Sudarshan and Mukunda, 1974). The precise number of constraints depends on the rank of (G_{AB}). We then select a part of the Q space where this rank is constant:

$$\operatorname{rank}[G_{AB}(Q)] = R; \qquad 0 < R \leq 4N \tag{8}$$

For our chosen R, the treatment is applied in the $(\dot{Q}, Q)$ region identified by equations (6) and (8).

When the rank of (G_{AB}) is R, there are $4N - R$ independent vectors $V_l(Q)$ such that

$$G_{BA} V_l^A = 0; \qquad l = 1, 2, \ldots, 4N - R \tag{9}$$

Therefore, the first $4N - R$ primary constraints can be written as

$$P_A V_l^A(Q) = 0 \tag{10}$$

At this point, one has already introduced all the constraints that are needed to take into account the functional interdependence of the P_A with respect to the variables S^B [equation (7a)]. In fact

$$\partial P_A / \partial S^B = G_{AB} \tag{11}$$

The next step is to examine equation (7b), which is also not invertible (that is, the $\dot{Q}$ variables cannot be solved as functions of the S variables).

It is readily seen that equation (7b) implies the identity

$$S^A G_{AB} S^B = 1 \tag{12}$$

In terms of the P variables, this is equivalent to the primary constraint

$$P_A H^{AB}(Q) P_B = 1 \tag{13}$$

where (H^{AB}) is any of the matrices such that

$$G_{AC} H^{CD} G_{DB} = G_{AB}, \qquad A, B, C, D = 1, 2, \ldots, 4N \tag{14}$$

Clearly, the relationship expressed by (13) has the structure of an Einstein condition. (That is, a quadratic expression of the momenta being equal to a constant.) Furthermore, the same equation can be considered as a straightforward generalization of well-known two-particle cases. In such cases, the choice for G_{AB} is

$$G_{AB}(Q) = G_{AB}^{\text{free}}(Q)\, V(Q) \tag{15}$$

where G_{AB}^{free} is suitable for free particles, and V plays the role of the "multiplicative potential" (Kalb and Van Alstine, 1976).

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